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Effect of magnetic field and buoyancy on onset of Marangoni convection

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INTRODUCTION

THIS IS an extension of our previous paper on the MHD Marangoni convection [1] in which we have added the buoyancy effect. The instability problem of buoyancy convection and Marangoni convection in a horizontal layer of an electrically conducting liquid, the top surface of which is free and the bottom is a rigid wall, is studied. The effects of magnetic field and aspect ratio of the liquid layer on the onset of buoyancy convection and the effect of buoyancy on the onset of Marangoni convection are made clear.

ANALYSIS

The governing equations are the same as those in ref. [1] except the momentum equation, which is written as below for the case of vertical magnetic field ($\delta = 0$)

$$\Delta^2 V_Z + Ra\,\Delta_{II}\theta - M^2\frac{\partial^2 V_Z}{\partial Z^2} = 0 \tag{1}$$

where Δ_{II} is the two-dimensional Laplace operator with regard to a horizontal plane.

Onset of natural convection in an infinite liquid layer

The momentum and the energy equations can be expressed by the harmonic analysis as follows [1–5]:

$$[(D^2-k^2)^2 - M^2D^2]F(Z) - Ra\,k^2G(Z) = 0 \tag{2}$$

$$F(Z) + (D^2-k^2)G(Z) = 0. \tag{3}$$

Functions $F(Z)$ and $G(Z)$ can be expanded, respectively, with a series of trial functions f_i and g_i

$$F(Z) = \alpha_i f_i \tag{4}$$

$$G(Z) = \beta_i g_i \tag{5}$$

where

$$f_i = (1-Z)Z^{i+1} \tag{6}$$

$$g_i = Z^i \qquad \text{for } Bi \neq \infty$$
$$= (1-Z)Z^i \qquad \text{for } Bi = \infty. \tag{7}$$

The criteria for the onset of Marangoni convection and buoyancy convection are obtained by applying the Galerkin method as follows:

onset of Marangoni convection

$$\det\left[\frac{1}{Ma}I + k^2(A_{11} - M^2B_{11} - k^2\,Ra\,B_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1}A_{21}\right] = 0; \tag{8}$$

onset of buoyancy convection

$$\det\left[\frac{1}{Ra}I - k^2(A_{11} - M^2B_{11})^{-1}B_{12}A_{22}^{-1}A_{21}\right] = 0 \tag{9}$$

where

$$A_{11} = \int_0^1 D^2f_mD^2f_i\,\mathrm{d}Z + 2k^2\int_0^1 Df_mDf_i\,\mathrm{d}Z + k^4\int_0^1 f_mf_i\,\mathrm{d}Z$$

$$A_{12} = Df_m(1)g_i(1)$$

$$A_{21} = \int_0^1 g_mf_i\,\mathrm{d}Z$$

$$A_{22} = Bi\,g_m(1)g_i(1) + \int_0^1 Dg_mDg_i\,\mathrm{d}Z + k^2\int_0^1 g_mg_i\,\mathrm{d}Z \quad \text{for } Bi \neq \infty$$

$$= -\int_0^1 g_mD^2g_i\,\mathrm{d}Z - k^2\int_0^1 g_mg_i\,\mathrm{d}Z \quad \text{for } Bi = \infty$$

$$B_{11} = -\int_0^1 Df_mDf_i\,\mathrm{d}Z$$

$$B_{12} = \int_0^1 f_mg_i\,\mathrm{d}Z.$$

Equations (8) and (9) are eigenvalue equations, where $1/Ma$ and $1/Ra$ are the eigenvalues. The Marangoni number and the Rayleigh number corresponding to the maximum eigenvalues become the critical Marangoni number and the critical Rayleigh number for a given wave number, respectively. The minimum critical Marangoni number and the minimum critical Rayleigh number when the wave number is varied are the actual critical Marangoni number and the critical Rayleigh number, respectively.

Onset of buoyancy convection in an infinite liquid layer

The critical Rayleigh numbers and the critical wave numbers obtained by solving equation (9) are listed in Table 1.

Chandrasekhar [2] has obtained relations between Ra_c, k_c and M^2 for the case of $Bi = \infty$ and $M^2 \to \infty$ by asymptotic analysis. The results are given as

$$Ra_c \to \pi^2M^2 \tag{10}$$

$$(M^2 \to \infty)$$

$$k_c \to \left[\frac{1}{2}\pi^4M^2\right]^{1/6}. \tag{11}$$

Table 1. Critical Rayleigh number and critical wave number for infinite liquid layer

	Bi							
M^2	0	0.01	0.1	1.0	10.0	100.0	1000.0	∞
0	$Ra_c = 668.998$	670.381	682.360	770.570	989.492	1085.90	1099.12	1100.65
	$k_c = 2.086$	2.089	2.116	2.293	2.589	2.672	2.681	2.682
0.1	671.145	672.529	684.524	772.842	992.040	1088.58	1101.82	1103.35
	2.088	2.091	2.119	2.295	2.592	2.675	2.684	2.685
1	690.373	691.772	703.899	793.180	1014.83	1112.54	1125.95	1127.50
	2.109	2.112	2.140	2.318	2.616	2.699	2.709	2.710
5	773.953	775.415	788.086	881.340	1113.18	1215.81	1229.93	1231.55
	2.195	2.198	2.227	2.408	2.714	2.800	2.809	2.810
10	874.862	876.393	889.662	987.314	1230.61	1338.84	1353.76	1355.49
	2.288	2.292	2.321	2.506	2.819	2.907	2.917	2.919
20	1067.89	1069.54	1083.81	1188.92	1452.10	1570.24	1586.60	1588.49
	2.443	2.447	2.477	2.668	2.992	3.084	3.095	3.096
50	1604.30	1606.19	1622.64	1744.16	2053.22	2195.24	2215.11	2217.41
	2.773	2.776	2.807	3.008	3.352	3.451	3.463	3.464
100	2424.90	2427.07	2445.84	2585.65	2949.81	3122.28	3146.72	3149.56
	3.128	3.132	3.163	3.368	3.729	3.836	3.849	3.850
200	3941.6	3944.1	3965.8	4129.2	4571.2	4790.1	4821.6	4825.3
	3.578	3.582	3.611	3.816	4.192	4.309	4.324	4.325
500	8112	8116	8141	8344	8935	9251	9298	9304
	4.319	4.322	4.348	4.545	4.946	5.060	5.078	5.080
1000	1.459×10^4	1.460×10^4	1.463×10^4	1.485×10^4	1.562×10^4	1.604×10^4	1.611×10^4	1.612×10^4
	4.99	4.99	5.02	5.18	5.59	5.73	5.75	5.75
2000	2.691×10^4	2.692×10^4	2.695×10^4	2.725×10^4	2.818×10^4	2.880×10^4	2.887×10^4	2.889×10^4
	5.74	5.75	5.79	5.95	6.33	6.48	6.50	6.51
5000	6.210×10^4	6.212×10^4	6.215×10^4	6.248×10^4	6.379×10^4	6.477×10^4	6.507×10^4	6.509×10^4
	6.94	6.94	6.98	7.09	7.46	7.62	7.65	7.66
10000	1.191×10^5	1.191×10^5	1.191×10^5	1.193×10^5	1.209×10^5	1.223×10^5	1.225×10^5	1.226×10^5
	7.96	7.96	7.97	8.06	8.43	8.63	8.65	8.65

The critical Rayleigh number becomes independent of the Biot number and approaches the relation expressed by equation (10) with increasing Hartmann number, and relation (11) gives the upper limit of the wave number when M^2 is large.

Effect of buoyancy on onset of Marangoni convection

The criterion of combined instability induced both by buoyancy and by surface tension is obtained by solving equation (8).

If the flow pattern of the Marangoni convection at a marginal state is exactly the same as that of the buoyancy convection, as Nield [4] has pointed out, the instability curve must satisfy the linear relation expressed as

$$\frac{Ma}{Ma_c} + \frac{Ra}{Ra_c} = 1. \tag{12}$$

The actual instability curve deviates from the linear relation (12) as either the Hartmann number or the Biot number increases. In fact, the differences of the wave numbers between Marangoni convection and buoyancy convection become large for large Bi and M^2. For example

$$k_c(\text{buoyancy}) = 2.086,\ k_c(\text{Marangoni}) = 1.993 \text{ for } Bi = 0,\ M^2 = 0$$

$$k_c(\text{buoyancy}) = 2.681,\ k_c(\text{Marangoni}) = 3.010 \text{ for } Bi = 10^3,\ M^2 = 0$$

$$k_c(\text{buoyancy}) = 4.99,\ k_c(\text{Marangoni}) = 4.745 \text{ for } Bi = 0,\ M^2 = 10^3$$

$$k_c(\text{buoyancy}) = 5.75,\ k_c(\text{Marangoni}) = 12.334 \text{ for } Bi = 10^3,\ M^2 = 10^3.$$

Onset of natural convection in circular cylindrical containers

The convective instability problem in a circular cylindrical container is considered. The liquid layer is subjected to a vertical temperature gradient and a vertical magnetic field, the side wall being thermally insulated. The aspect ratio A is defined as the ratio of the radius of the container to the depth of the liquid layer.

The perturbation equations are analyzed directly in the case of the finite liquid layer by assuming that the steady two-dimensional concentric rolls are generated at a marginal state [1, 6]. The perturbations V_Z and θ are expanded with a series of trial functions F_{ij} and G_{ij}, respectively

$$V_Z = \alpha_{ij} F_{ij} \tag{13}$$

$$\theta = \beta_{ij} G_{ij} \tag{14}$$

$$F_{ij} = b_{0,i}(R/A) f_j(Z) \tag{15}$$

$$G_{ij} = J_0(\mu_i R/A) g_j(Z) \tag{16}$$

where $b_{0,i}(R/A)$ is as it appeared in ref. [1] and $f_j(Z)$ and $g_j(Z)$ are given by equations (6) and (7).

The criteria for the onset of Marangoni convection and buoyancy convection are:

onset of Marangoni convection

$$\det\left[\frac{1}{Ma}I + (A_{11} - B_{11} - M^2 C_{11} - Ra\, A_{12} A_{22}^{-1} A_{21})^{-1} B_{12} A_{22}^{-1} A_{21}\right] = 0; \tag{17}$$

onset of buoyancy convection

$$\det\left[\frac{1}{Ra}I - (A_{11} - B_{11} - M^2 C_{11})^{-1} A_{12} A_{22}^{-1} A_{21}\right] = 0 \tag{18}$$

where

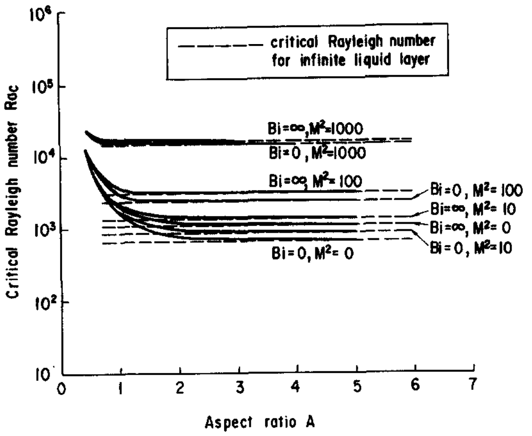

FIG. 1. Dependence of critical Rayleigh number on aspect ratio.

$$A_{11} = \int_v \Delta F_{mn} \Delta F_{ij} \,\mathrm{d}v$$

$$A_{12} = \int_v F_{mn} \Delta G_{ij} \,\mathrm{d}v$$

$$A_{21} = \int_v G_{mn} F_{ij} \,\mathrm{d}v$$

$$A_{22} = \int_v \nabla G_{mn} \nabla G_{ij} \,\mathrm{d}v$$

$$-Bi \int_{Z=1} G_{mn} G_{ij} \,\mathrm{d}s \quad \text{for } Bi \neq \infty$$

$$= \int_v G_{mn} \Delta G_{ij} \,\mathrm{d}v \quad \text{for } Bi = \infty$$

$$B_{11} = \int_{R=A} \frac{\partial F_{mn}}{\partial R} \Delta F_{ij} \,\mathrm{d}s$$

$$B_{12} = \int_{Z=1} \frac{\partial F_{mn}}{\partial Z} \Delta_{\mathrm{II}} G_{ij} \,\mathrm{d}s.$$

Onset of buoyancy convection in a container

The dependence of the critical Rayleigh number on the aspect ratio is shown in Fig. 1 for combinations of $Bi = 0$, ∞ and $M^2 = 0$, 10, 100 and 1000. The critical Rayleigh

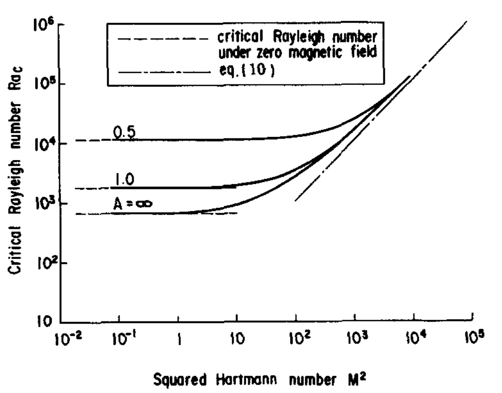

FIG. 2. Dependence of critical Rayleigh number on squared Hartmann number.

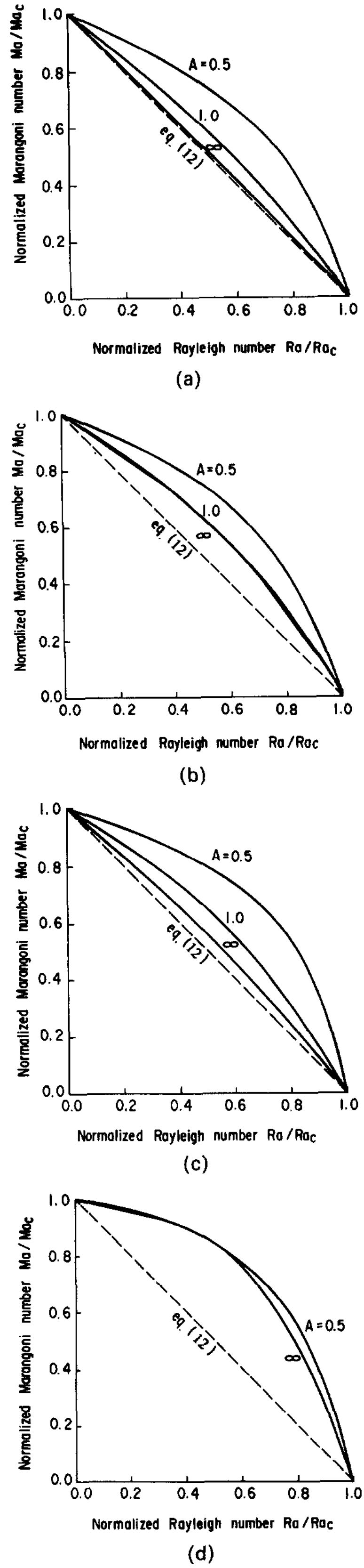

FIG. 3. Normalized instability curve for a finite liquid layer in a container: (a) $Bi = 0$, $M^2 = 0$; (b) $Bi = 0$, $M^2 = 1000$; (c) $Bi = 100$, $M^2 = 0$; (d) $Bi = 100$, $M^2 = 1000$.

numbers under zero magnetic field are indicated by the broken lines.

The critical Rayleigh number increases with decreasing aspect ratio because of the effect of the side wall, although the effect becomes small as both the Hartmann number and the Biot number increase.

Figure 2 shows the dependence of the critical Rayleigh number on the squared Hartmann number for the case of $Bi = 0$. As the magnetic field is intensified, the effect of the aspect ratio becomes small and the critical Rayleigh number in a finite liquid layer approaches that of an infinite layer, which follows equation (10) for large M^2. A similar tendency is also obtained for different Biot numbers.

Effect of buoyancy on onset of Marangoni convection

The normalized instability curves are obtained on the basis of both our previous study [1] and the results given in the previous section. The curves are shown in Fig. 3 for combinations of $Bi = 0$, 100 and $M^2 = 0$, 1000.

As the aspect ratio is reduced, the deviation from the linear relation (12) becomes large and the instability curve approaches a rectangular shape, since the difference in flow patterns between the Marangoni convection and the buoyancy convection becomes large when the aspect ratio is small. However, the effect of the aspect ratio on the instability curve becomes small as the Hartmann number increases.

CONCLUSION

The onset of buoyancy convection and Marangoni convection in a horizontal layer of an electrically conducting liquid has been studied theoretically and the following results have been obtained.

(1) The critical Rayleigh number increases as the Hartmann number and the Biot number at the free surface increase and as the aspect ratio of the liquid layer decreases.

(2) The effect of the aspect ratio on the critical Rayleigh number and the flow pattern vanishes when the Hartmann number is sufficiently large.

(3) The instability curves for the onset of combined convection driven both by buoyancy and by surface tension forces deviate greatly from the linear relation $Ma/Ma_c + Ra/Ra_c = 1$ as the Hartmann number and the Biot number increase and as the aspect ratio decreases.

(4) The effect of the aspect ratio on the onset of combined convection vanishes for sufficiently large Hartmann numbers.

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Viscous dissipation effects in buoyancy induced flows

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1. INTRODUCTION

THE GENERAL equations of fluid transport for buoyancy induced flows are

$$\frac{D\rho}{D\tau} = -\rho\nabla\cdot\bar{V} \tag{1}$$

$$\frac{\rho D\bar{V}}{D\tau} = \rho\bar{g} - \nabla p + \mu\nabla^2\bar{V} + \frac{1}{3}\mu\nabla(\nabla\cdot\bar{V}) \tag{2}$$

$$\rho c_p\frac{Dt}{D\tau} = \nabla\cdot k\nabla t + \beta T\frac{Dp}{D\tau} + \mu\Phi + q'''. \tag{3}$$

The viscous dissipation energy effect is $\mu\Phi$, where Φ in Cartesian coordinates, see, e.g. refs. [1, 2], is given by

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right] - \frac{2}{3}(\nabla\cdot\bar{V})^2. \tag{4}$$

One of the common assumptions made in deriving a simpler form of these equations for boundary layer flows is that the viscous dissipation effects may often be neglected. Order of magnitude arguments are used to identify the necessary conditions. A common conclusion is that, for $g\beta L/c_p = R_0 < 1$, the viscous dissipation effects may be neglected for $Pr < 1$, see, e.g. ref. [2].